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A quadrangle with side lengths and area that are integers and inscribed in a circle

Mathematics Group

1.Introduction.

Triangles with three consecutive integers of side lengths and integer area have already been generalized. We started this research because we wondered what kind of regularity exists in the case of quadrilaterals with integer side lengths and areas. The aim of this research is to discover the connection between the world of geometry and the world of integers.

In the case of a quadrangle, determining only the lengths of the sides does not make the quadrangle a single one, so we will restrict our discussion to the case of a quadrangle inscribed in a circle. In this case, it is possible to use Brahmagupta's formula to find the area.

[Brahmagupta's formula]

In a quadrangle inscribed in a circle whose four sides have lengths a, b, c, d,
if $t = \frac{a+b+c+d}{2}$, the area S of the quadrangle is

$$S = \sqrt{(t-a)(t-b)(t-c)(t-d)}$$

2.Definition.

・**Brahmagupta quadrangle**…A quadrangle that is inscribed in a circle and whose side lengths and areas are both integers.

3.Result.

In Brahmagupta's quadrangle, we have shown the following.

① There is no **Brahmagupta** quadrandle whose side lengths are consecutive 4 integers.

② There is no Brahmagupta quadrangle with a constant difference in length between the four sides. (① was made more general.)

③ We can consider how to construct a Brahma-Gupta quadrangle, and by doing so, we can represent all Brahma-Gupta quadrangles.

Proof of ①.

(Proof) Let the lengths of the sides be $n-1, n, n+1, n+2$.

From $\frac{(n-1)+n+(n+1)+(n+2)}{2} = 2n+1$, the area S is

$$S = \sqrt{\{(2n+1)-(n-1)\}\{(2n+1)-n\}\{(2n+1)-(n+1)\}\{(2n+1)-(n+2)}$$
$$= \sqrt{(n-1)n(n+1)(n+2)}$$

$= \sqrt{(n^2+n+1)^2-1}$

From the above, S^2 =(square number)−1, which shows that such (n, S) does not exist.

Proof of ②：See Appendix.

(outline) In the process of proof, it is necessary to divide the cases according to the evenness and oddness of the tolerance of the equi-difference sequence.

In addition, the following are used in the proof process.

［Primitive Pythagorean numbers]…A pair of natural numbers (a, b, c) that are prime to each other satisfying $a^2+b^2=c^2$.

From the fact that a and b are even-odd and different and from the symmetry of a and b, b is assumed to be even.

For any pair of primitive Pythagorean numbers (a, b, c), there exists some m, n (m, n are coprime natural numbers, $m > n$, m, n even and odd are distinct) satisfying the following equation.

$$\begin{cases} a = m^2 - n^2 \\ b = 2mn \\ c = m^2 + n^2 \end{cases}$$

[Proof by infinite descent]…A backward reasoning based on the fact that there always exists a minimum value in the set of natural numbers.

For any natural number m satisfying condition A, there exists a natural number n satisfying condition A such that $m > n$.

Condition A is contradictory because there is no minimum value in the set of natural numbers satisfying condition A.

Proof of ③:See appendix.

(outline) In the figure on the right, △PAD∽△PCB, since BC and AD are integers, for any Brahmagupta quadrangle, we can construct △PBC whose side lengths and areas are rational numbers.

If we expand this triangle by a rational number, we obtain a triangle whose side lengths and areas are integers (Heron's triangle), and since the side lengths of Heron's triangle are generalized, we can construct any Brahmagupta's quadrangle.

4. Future Prospect.

・Re-examine the Brahmagupta quadrangle with ③ and discuss other properties of the quadrangle using a computer or other means.

・We will discuss why this method of proof is applicable.

5. References.

Well-Known Problems and Theorems in Mathematics ヘロンの三角形 https://wkmath.org/heron-f.html

Wikipedia Heronian triangle https://en.wikipedia.org/wiki/Heron https://en.wikipedia.org/wiki/Heronian_triangle ian_triangle

Proof of ②.

In i), the following Lemma is used, which is shown first.

[Lemma] There does not exist a pair of primitive Pythagorean numbers (b,k,x) and $(3b,l,x)$ such that b is even.

(Proof) Assuming that the above pairs exist, consider the one with the smallest x among them.

From the properties of primitive Pythagorean numbers, there must exist K,L,M,N such that $\begin{cases} b=2KL \\ k=K^2-L^2 \\ x=K^2+L^2 \end{cases}$ and $\begin{cases} 3b=2MN \\ l=M^2-N^2 \\ x=M^2+N^2 \end{cases}$ (Note that K,L,M,N are natural numbers, K and L, M and N are coprime, and the even-oddness of K and L, M and N is different.)

Therefore, $\begin{cases} 3KL=MN \ldots\ldots ③ \\ K^2+L^2=M^2+N^2 \ldots\ldots ④ \end{cases}$ must hold. We ignore the relationship between M and N and assume that M is a multiple of 3 from symmetry.

Let $K=AB, L=CD, M=3AC, N=BD$(A,B,C,D are natural numbers and B and D are not multiples of 3). Here, if any two of $A,B,C,$ or D are not coprime, it contradicts the fact that K and L, M and N are coprime. Thus, any two of $A,B,C,$ and D arecoprime. Moreover, since the even-oddness of K and L, M and N is different only one of $A,B,C,$ or D is even.

From ④, $(AB)^2+(CD)^2=(3AC)^2+(BD)^2$, therefore $A^2(B^2-9C^2)=D^2(B^2-C^2)$ and

Since A and D are coprime, we have $\begin{cases} B^2-C^2=tA^2 \ldots\ldots ⑤ \\ B^2-9C^2=tD^2 \ldots\ldots ⑥ \end{cases}$ (t is an integer).

Since $t=\pm(B^2-C^2,B^2-9C^2)=\pm(B^2-C^2,8)$ ($\because B$ and C are coprime), t is a divisor of 8.

From ⑥, $B^2\equiv tD^2(mod3)$, and

since B and D are not multiples of 3, $B^2\equiv D^2\equiv 1(mod3)$, so $t\equiv 1(mod3)$.

If either A or D is even when $t=-2,4$, then B^2-C^2 and B^2-9C^2 are both multiples of 8, which means that both A and D are even, contradicting the fact that A and D are coprime. Thus, both A and D are odd, but from B^2-C^2=(even), the even-oddness of B and C is different, but if both B and C are even, it contradicts that B and C are coprime, and if B and C are odd, it contradicts that one of A,B,C and D is even.

From the above, $t=1,-8$. Note that B and D are not multiples of 3.

If $t=1$, $\begin{cases} B^2=A^2+C^2 \\ B^2=D^2+(3C)^2 \end{cases}$ (B and C, B and $3C$ are coprime), the greatest common divisor of B^2-C^2 and 8 is 1, and since A is odd, C is even.

Also, $B\leq AB=K<K^2+L^2=x$.

If $t=-8$, $\begin{cases} D^2=C^2+A^2 \\ D^2=B^2+(3A)^2 \end{cases}$ (A and $D, 3A$ and D are coprime), B^2-C^2 is more than even, B and C are odd, A is even. Also, $D \leq CD = L < K^2+L^2 = x$.
Therefore, both cases contradict the minimality of x.
Thus, it is shown by the infinite descent method.

The following is a proof that there does not exist a Brahmagupta quadrangle with a constant difference in length between the four sides.
(Proof) Hereafter, we denote the greatest common divisor of a and b by (a,b).
Let the lengths of the sides be $d(n-a), dn, d(n+a), d(n+2a)$ (n, a, d are integers, n and a are disjoint).
i) When a is even :
$S = d^2\sqrt{(n-a)n(n+a)(n+2a)} = d^2\sqrt{(n^2+an)(n^2+an-2a^2)}$ and
$(n^2+an, n^2+an-2a^2) = (n^2+an, -2a^2) = (n^2+an, 2)$($\because$ n, a are coprime).
a is even, and n is odd ($\because$ n and a are coprime), then n^2+an is odd.
Therefore $n^2+an-2a^2$ and n^2+an are coprime and odd.

Since S is an integer, there exists an odd pair (k,l) such that $\begin{cases} n^2+an=k^2 \ldots\ldots ① \\ n^2+an-2a^2=l^2 \ldots\ldots ② \end{cases}$.

a is even, so $a=2b$(b is an integer) and $x=(n+b)^2$.

Then from ①, $k^2+b^2=x^2 \ldots\ldots ①'$, from ②, $l^2+(3b)^2=x^2 \ldots\ldots ②'$ and

with ①', x and b are coprime, it is impossible that x cannot be a multiple of 3.
Therefore, since x and $3b$ are also coprime, the pairs of integers $(b,k,x), (3b,l,x)$ are pairs of primitive Pythagorean numbers since two of the three numbers are coprime.
Since k is odd, b is even from $mod4$ in ①.
This does not exist from the lemma.
ii)When a is odd:
$n^2+an=n(n+a)$ is even.
Therefore, since the greatest common divisor of n^2+an and $n^2+an-2a^2$ is 2, and since S is an integer,
there exist k, l(k, l are natural numbers) such that $\begin{cases} n^2+an=2k^2 \ldots\ldots ① \\ n^2+an-2a^2=2l^2 \ldots\ldots ② \end{cases}$.
Since n and a are coprime, a and l are coprime from ②.
Moreover, if ② is regarded as a quadratic equation for n and let D be the discriminant equation, D must be a square number. $D = a^2-4(-2a^2+2l^2) = 9a^2+8l^2$.

Since the remainder of the square number divided by 3 is 0 or 1, l must be a multiple of 3.

From (①+②) ÷2, (①−②)÷2, $\begin{cases} n^2 + an - a^2 = k^2 + l^2 \cdots\cdots ③ \\ \qquad a^2 = k^2 - l^2 \cdots\cdots ④ \end{cases}$

From ④, since $a^2 + l^2 = k^2$ and a and l are coprime, (a, l, k) are primitive Pythagorean numbers.

Therefore, since a is odd, we have $\begin{cases} a = M^2 - N^2 \\ l = 2MN \\ k = M^2 + N^2 \end{cases}$ (M and N are natural numbers, M and N are coprime, the even-oddness of M and N is different.).

Since l is a multiple of 3, only one of M and N is a multiple of 3.

Substituting for ③, $n^2 + (M^2 - N^2)n - (M^2 - N^2) = (M^2 + N^2)^2 + 4M^2N^2$

To rearrange, $n^2 + (M^2 - N^2)n - 2(M^4 + 2M^2N^2 + N^2) = 0$

Let D' be this discriminant equation.

$$D' = (M^2 - N^2)^2 + 8(M^2 + 2M^2N^2 + N^4) = (3M^2 + 2MN + 3N^2)(3M^2 - 2MN + 3N^2)$$

Let $A = 3M^2 + 2MN + 3N^2, B = 3M^2 - 2MN + 3N^2 (> 0)$.

$A - B = 4MN$, and since A and B are prime to each other with $2, M,$ and N, A and B are prime to each other.

Given that D' is a square number, we can write $A = s^2$ and $B = t^2$ (s and t are natural numbers).

That is, $\begin{cases} 3M^2 + 2MN + 3N^2 = s^2 \cdots\cdots ⑤ \\ 3M^2 - 2MN + 3N^2 = t^2 \cdots\cdots ⑥ \end{cases}$

From the symmetry of M and N, suppose that M is a multiple of 3 and N is not a multiple of 3. Let $M = 3m$ (m is an integer).

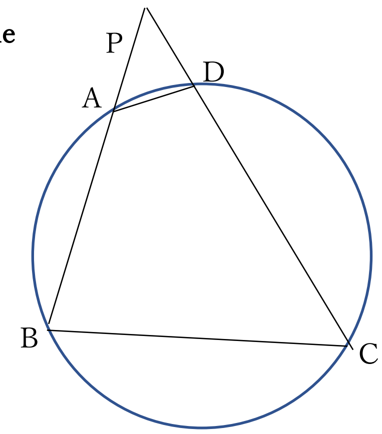
From ⑤ and ⑥, $\begin{cases} 27m^2 + 6mN + 3N^2 = s^2 \\ 27m^2 - 6mN + 3N^2 = t^2 \end{cases}$

Therefore, s and t are multiples of 3 and we have $s = 3s', t = 3t'$ (s' and t' are integers).

From ⑤+⑥, $3 \cdot 2(M^2 + N^2) = 3^2(s'^2 + t'^2)$, which is a contradiction since $M^2 + N^2$ is not a multiple of 3.

Therefore, when a is odd, such a quadrangle does not exist.

How to construct a quadrangle in Brahmagupta's quadrangle



When quadrangle ABCD is a rectangle with four sides of integer length, it is obvious that the rectangle ABCD is a quadrangle of Brahmagupta, so we will consider the following non-rectangular cases. For the Brahmagupta quadrangle ABCD, given that the sum of the opposite angles of a quadrangle inscribed in a circle is 180° , there exist adjacent angles of quadrangle ABCD that are both acute angles.

From the symmetry, let B and C be the two angles. Let P be the intersection of half lines BA and CD.

From △PAD∽△PCB, let the similarity ratio be $1{:}s = AD{:}CB$, then s is a rational number when AD and BC are rational numbers.

Therefore, △PAD is a triangle with three rational lengths and rational areas. By expanding it by a rational factor, we obtain a triangle with three mutually prime integers in length and an integer area (Heron's triangle) is obtained.

The side lengths of Heron's triangle are $mn > h^2 \geq \frac{m^2 n}{2m+n}$, using the mutually prime natural numbers m, n, h such that $m \geq n$,$n(m\text{^}2 + h\text{^}2\ ), m(n\text{^}2 + h\text{^}2\ ), (m+n)(mn - h\text{^}2)$.

This can be used to find the pair of sides of the Brahmagupta quadrangle.

Specifically, any Brahmagupta quadrangle can be constructed by determining Heron's triangle and s above and multiplying them by rational numbers.