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Menelaus' Theorem in n Dimensions

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1. Background

Menelaus'theorem is a theorem with a triangle and a line, and is a theorem about ratios of edges. From prior research, there is a theorem which is defined with a tetrahedron and a plane and is about ratios of areas. Then, we supposed and proved Menelaus'theorem in n dimensions.

2. Definition

1, n-simplex

a n-dimensional polytope which is the convex hull of its n+1vertice(a generalization of a triangle or tetrahedron to arbitrary dimensions)

2, n-hyperplane

a subspace whose dimension is one less than that of its ambient space(if a space is 3-dimensional then its hyperplanes are the 2-dimensional planes, while if the space is dimensional, its hyperplanes are the 1-dimensional lines)

3, n-hypervolume

the volume of a dimensional polytope

4, n-face

When two n-simplexes share one vertice, and the one includes the other, we call the latter one n-face

3. Hypervolume of Hyperpyramid

Hyperpyramid is a generalisation of the normal pyramid to n dimension.
The base of the hyperpyramid is a (n-1)-polytope in a (n-1)-hyperplane.
Apex is a point which is located outside the hyperplane and gets connected to all the vertices of thepolytope. The height of the hyperpyramid is the distance from the apex to the base.
If
S=the area of the base of the hyperpyramid
h=the height of the hypermiramid

$V=\frac{Sh}{n}$.is proper.

4. Lemma

We are going to prove that
"hypervolume of n-simplex : hypervolume of n-face=

$$\prod_{i=1}^{n} a_i \ : \ \prod_{i=1}^{n} b_i$$

is valid by mathematical induction.
i)When n=1 prove the equation holds true.
ii)Assume n=k hold true.
When n=k+1 $a_{k+1}:b_{k+1}$=PH:QI
∴ hypervolume of (k+1)-simplex : hypervolume of

(k+1)-face= $a_{k+1}\prod_{i=1}^{k} a_i \ : \ b_{k+1}\prod_{i=1}^{k} b_i$

$$= \prod_{i=1}^{k+1} a_i \ : \ \prod_{i=1}^{k+1} b_i \qquad \blacksquare$$

5. Menelaus' theorem in n-D

Make a m-simplex A0A1A2…Am cut by (m-1)-hyperplane B1B2…Bm. Vertex Bk is a vertex located in segment A0Ak, and k which makes the distance between vertex Bk and hyperplane A1A2…Am minimum equals to m.

Make hyperplane at the intersection of hyperplane A1A2…Amand B1B2…Bm as C1C2…Cm-1. C_k is a vertex located as the intersection of ray B_kB_m and A_kA_m.

We define that s(C0C1C2…Cn) represent the hypervolume of the simplex C0C1C2…Cn. The formula below is valid with this notation.

$$\frac{s(A_mC_1C_2\cdots C_{m-1})}{s(A_!A_2A_3\cdots A_m)} \ \frac{s(A_!A_2A_3\cdots A_m)}{s(A_0B_1B_2\cdots B_{m-1})} \ \frac{s(B_1B_2B_3\cdots B_m)}{s(A_mC_1C_2\cdots C_{m-1})} = 1$$

Proof

From menelalus, $1 \leqq {}^{\forall}k \leqq m-1$, $\frac{A_mC_k}{A_mA_k} \cdot \frac{A_0A_k}{A_0B_k} \cdot \frac{B_mB_k}{B_mC_k}=1$ is valid.From this fomula and Lemmata,

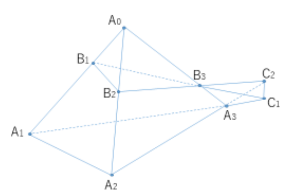
$$\frac{s(A_mC_1C_2\cdots C_{m-1})}{s(A_1A_2A_3\cdots A_m)} \cdot \frac{s(A_0A_1A_2\cdots A_{m-1})}{s(A_0B_1B_2\cdots B_{m-1})} \cdot \frac{s(B_1B_2B_3\cdots B_m)}{s(B_mC_1C_2\cdots C_{m-1})}$$

$$= \frac{\prod_{k=1}^{m-1} A_mC_k}{\prod_{k=1}^{m-1} A_mA_k} \cdot \frac{\prod_{k=1}^{m-1} A_0A_k}{\prod_{k=1}^{m-1} A_0B_k} \cdot \frac{\prod_{k=1}^{m-1} B_mB_k}{\prod_{k=1}^{m-1} B_mC_k}$$

$$= \prod_{k=1}^{m-1} \frac{A_mC_k}{A_mA_k} \cdot \frac{A_0A_k}{A_0B_k} \cdot \frac{B_mA_k}{B_mC_k}$$

$$= \prod_{k=1}^{m-1} 1$$

$$= 1$$



6. Future Prospect

・Ceva's theorem in *n*-D

・Menelous' Theorem is a theorem with a line and a triangle. It is also known that there are similar theorems with a line and a square, a line and a pentagon, and so on. We will prove these theorems in n-D.

7. Preferences

[1]　Sora Tazaki etc.(**Our Research**)
"'Oiras'*chou* Polytope Theorem'
Deciding Euler Characteristic of Polytopes "
[2]Koji Shiogai　「n 次元における n 次元単体と n 次元の面の超体積比の一般化」
[3] 平面図形の定理の空間への拡張 https://www.nagano-c.ed.jp/seiho/intro/risuka/kadaikenq/paper/2021/7_heimen.pdf
(2022.9.15)
[4]チェバ・メネラウスの定理に関する教材開発：n 角形への拡張
file:///C:/Users/21151/Downloads/29-0090.pdf
(2022.9.15)

8. Appedix

Figure1